Antipode formulas for pattern Hopf algebras

Raul Penaguiao and Yannic Vargas
San Francisco State University, California US
raulpenaguiao@sfsu.edu

Hopf monoids in species

a species in combinatorics is a mapping $h : I \mapsto \{\text{combinatorial structures on } I\}$
examples of species in combinatorics: graphs Gr with vertex set I, generalized permutahedra in $\mathbb{R}^I$, matroid structures on I, permutations of the set I
species can be endowed with a **product** and a **coproduct**. in a bimonoid these satisfy the following:

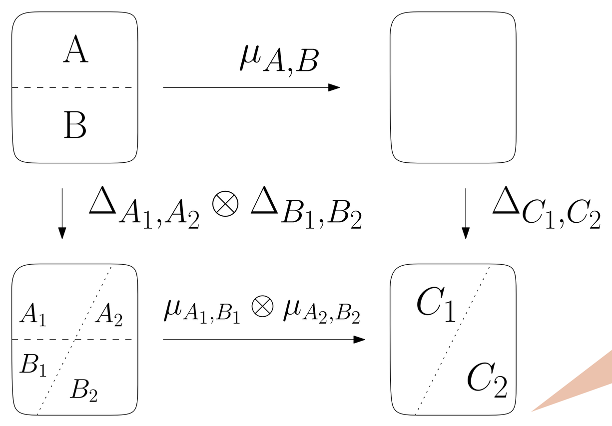


Fig. 1: The commutative diagram axiom

this is a commutative diagram. This means that compositions of the depicted maps are invariant of which path is taken. In this case, when working with Hopf monoids in combinatorial objects, the diagram claims that breaking objects appart and merging objects together can be done in any order one likes.

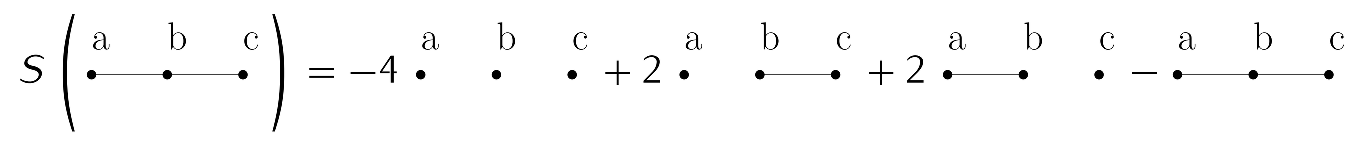
example of a bimonoidal structure on graphs G_1, G_2, G with vertex set V_1, V_2 and $V = V_1 \uplus V_2$

$$\mu_{V_1,V_2}(G_1, G_2) = \text{ disjoint union of } G_1, G_2$$
$$\Delta_{V_1,V_2} G = G|_{V_1} \otimes G|_{V_2}$$

Hopf monoid entails an **antipode**, described succintly in [HM12]:

$$S_I(G) = \sum_{f,o} (-1)^{c(f)} G|_f \,,$$

summing over all pairs of a flat f of G and an acyclic orientation o of $G/_f$, where $c(f)$ is the number of connected components of f. For instance,



permutation pattern Hopf algebras

square diagrams and one-line notation: permutations can be represented pictorically as

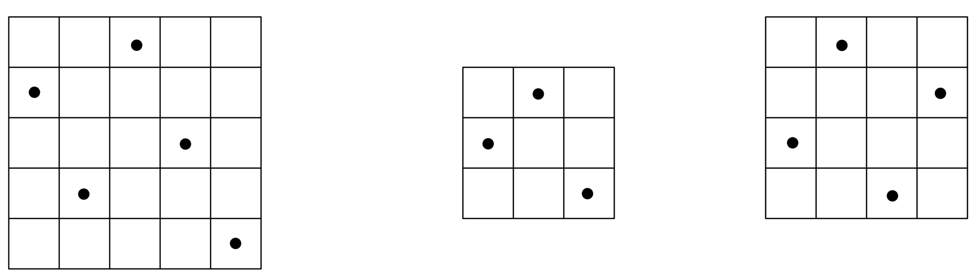


Fig. 2: Three permutations $\pi_1 = 42531$, $\pi_2 = 231$ and $\pi_3 = 2413$

restrictions of permutations arise when selecting the permutation corresponding to some columns

or instance, $\pi_1|_{1,3,5} = \pi_3$ and $\pi_1|_{1,2,3,4} = \pi_2$

counting restrictions that match a pattern τ defines permutation functions $\mathbf{p}_\tau : \mathfrak{S} \to \mathbb{Z}$

for example $\mathbf{p}_{\pi_2}(\pi_1) = 4$, $\mathbf{p}_{\pi_3}(\pi_1) = 1$ and $\mathbf{p}_{\pi_1}(\pi_3)$.

an unexpected relation on pattern functions arises on the **pointwise product**:

$$\mathbf{p}_{\tau_1}\mathbf{p}_{\tau_2} = \sum_\sigma \binom{\sigma}{\tau_1, \tau_2} \mathbf{p}_\sigma \,,$$

where the coefficients $\binom{\sigma}{\tau_1,\tau_2}$ are the **quasi-shuffle signatures (QSS)** of τ_1, τ_2 into σ

for example: $\mathbf{p}_{12}\mathbf{p}_1 = 3\mathbf{p}_{123} + 2\mathbf{p}_{132} + 2\mathbf{p}_{213} + \mathbf{p}_{231} + \mathbf{p}_{312} + 2\mathbf{p}_{12}$.
$\mathbf{p}_{12}(\pi_1)\mathbf{p}_1(\pi_1) = 3 \times 5 = 3 \times 0 + 2 \times 1 + 2 \times 1 + 4 + 2 \times 3$
$= 3\mathbf{p}_{123}(\pi_1) + \mathbf{p}_{132}(\pi_1) + 2\mathbf{p}_{213}(\pi_1) + \mathbf{p}_{231}(\pi_1) + \mathbf{p}_{312}(\pi_1) + 2\mathbf{p}_{12}(\pi_1)$.

this allows us to define the **permutation pattern algebra** $\mathcal{A}(\text{Per}) = \text{span}\{\mathbf{p}_\pi | \pi \in \mathcal{P}\}$.

theorem [V14]

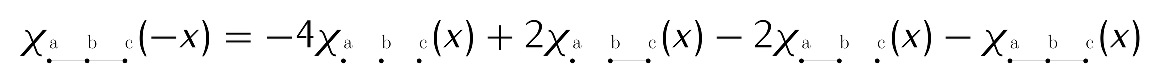
the permutation pattern algebra is free commutative generated by **Lyndon permutations**

Pattern Hopf algebras

most pattern Hopf algebras are free. It is conjectured that all such Hopf algebras are free, and there are some partial results towards this conjecture.

cancellation-free antipode formulas in Hopf algebras

reciprocity results in light of an antipode formula
an antipode on a Hopf algebra is a map $S : H \to H$ that is **anti-commutative**, preserves degree and is preserved under Hopf algebra morphisms. If $\chi : H \to \mathbb{K}[x]$ is a Hopf algebra morphism (think *chromatic polynomials* on graphs, for instance) then the antipode commutes with the morphism: $\chi_G(-x) = \chi_{S(G)}(x)$.



as a consequence we recover a long standing beautiful **reciprocity** result from [S73]

$$\chi_G(-1) = (-1)^{|V(G)|} \#\{\text{acyclic orientations of } G\}$$

this result can be extended to many cocommutative pattern Hopf algebras.

Negative result

the permutation pattern Hopf algebra $\mathcal{A}(\text{Per})$ has no non-trivial Hopf algebra morphism to $\mathbb{K}[x]$

cancellation-free antipode formula for permutations

an interlacing quasi-shuffle signature of σ from $\pi_1, \dots, \pi_k$ is a QSS of σ from $\pi_1, \dots, \pi_k$ such that no two patterns of consecutive permutations π_i, π_{i+1} arise diagonally in σ

$$\left[\begin{matrix}\sigma \\ \pi_1, \dots, \pi_k\end{matrix}\right] = \left|\{\text{interlacing QSS of } \sigma \text{ from } \pi_1, \dots, \pi_k\}\right|.$$

$$S(\mathbf{p}_\pi) = \sum_\sigma \left[\begin{matrix}\sigma \\ \pi_1, \dots, \pi_k\end{matrix}\right] \mathbf{p}_\sigma$$

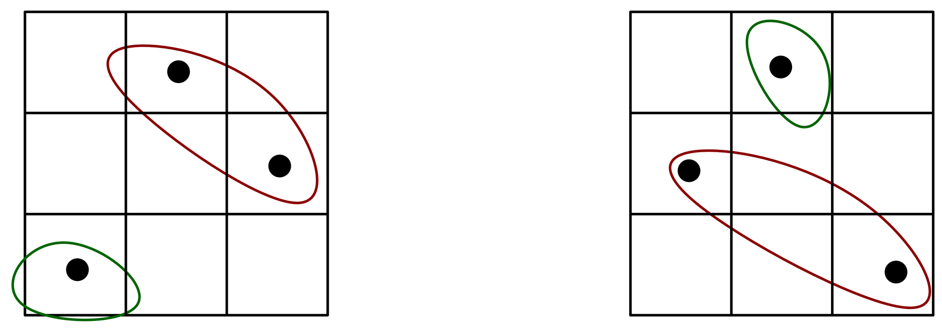


Fig. 3: **Left:** the only non-interlacing quasi-shuffle of 132 from 1, 21. **Right:** a QSS.

$$S(\mathbf{p}_{132}) = S(\mathbf{p}_{1\oplus 21}) = \mathbf{p}_{132} + \mathbf{p}_{213} + 2\mathbf{p}_{231} + 2\mathbf{p}_{312} + 3\mathbf{p}_{321} + 2\mathbf{p}_{21}.$$

this formula also works for other pattern algebras, like **packed words**

other pattern algebras - parking functions

n - parking functions are sequences $a_1 \dots a_n$ of length n containing numbers in $\{1, \dots, n\}$ such that after reordering in increasing order we have $a_i \leq i$.
there are $(n+1)^{n-1}$ parking functions.

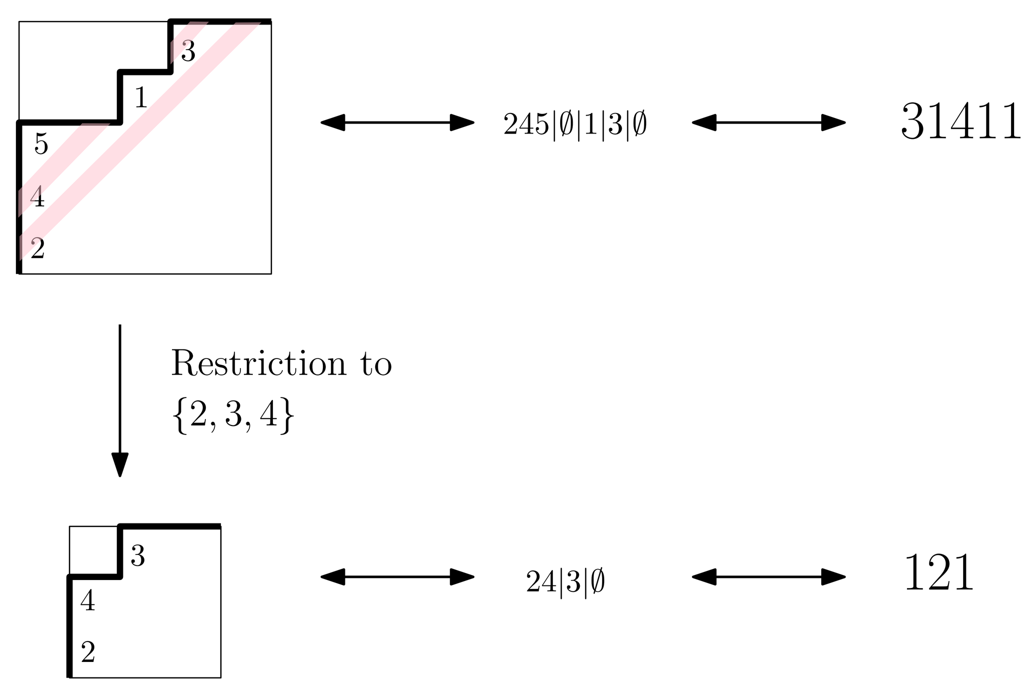


Fig. 4: A correspondence between parking functions and labelled Dyck paths allows us to see parking functions with labelling set I

with this we can count patterns in parking functions

$$\mathbf{p}_{11}(131) = 1, \quad \mathbf{p}_{21}(113) = 0, \quad \mathbf{p}_{21}(131) = 1$$

Parking functions Hopf algebras

parking functions arise as Hopf algebras in [NT03], here we present a new Hopf algebra structure in parking functions

Bibliography

[HM12] Humpert, Brandon, and Jeremy L. Martin. "The incidence Hopf algebra of graphs." SIAM Journal on Discrete Mathematics 26.2 (2012): 555-570.
[NT03] Novelli, Jean-Christophe, and Jean-Yves Thibon. "A Hopf algebra of parking functions." arXiv preprint math/0312126 (2003).
[PV22] R.P. and Y. Vargas, Antipode formulas for pattern Hopf algebras, to appear.
[P22] Penaguiao, Raul. "Pattern Hopf Algebras." Annals of Combinatorics (2022): 1-47.
[S73] Stanley, Richard P. "Acyclic orientations of graphs." Discrete Mathematics 5.2 (1973): 171-178.
[V14] Vargas, Yannic. "Hopf algebra of permutation pattern functions." Discrete Mathematics and Theoretical Computer Science (2014).