Probability 2

Exercise sheet nb. 5

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Due until: 22th October at 5 p.m.

Exercise 1 (3 points). Fix $m > k > 0$. Let $\{X_n\}_{n\geq 0}$ be the simple random walk on $\mathbb{Z}$ starting at k, that is we consider i.i.d. random variables $\{Y_n\}_{n\geq 1}$ so that $\mathbb{P}[Y_1 = 1] = \mathbb{P}[Y_1 = -1] = 0.5$, and let $X_0 = k$, $X_{n+1} = X_n + Y_{n+1}$ for $n \geq 0$.

Let T be the stopping time given by

$$T = \min_{n\geq 0}\{X_n = m \text{ or } X_n = 0\}.$$

1. Show that T is a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, \dots Y_n)$.
2. Does $\{X_{n\wedge T}\}_{n\geq 0}$ converge a.s.?
3. Show that T is almost surely finite. What is the limit of $\mathbb{E}[X_{n\wedge T}]$? Compute $\mathbb{P}[X_T = 0]$.

Exercise 2 (1 points). For each $n \geq 1$, let X_n be a random variable with

$$\mathbb{P}(X_n = n) = \tfrac{1}{n} = 1 - \mathbb{P}(X_n = 0).$$

Show that the family $(X_n)_{n\geq 1}$ satisfies $\sup_{n\geq 1} \mathbb{E}(|X_n|) < \infty$ but is not uniformly integrable.

Exercise 3 (5 points). Let $\{X_n\}_{n\geq 0}$ be a sequence of integrable **nonnegative** r.v. with $X_n \to X$ a.s. In this exercise, we show that uniform integrability is a **necessary and sufficient condition** to have $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

1. Assume that X_n is u.i. Show that $\mathbb{E}[X_n] \to \mathbb{E}[X]$.
2. Define $X_n^{(\alpha)} = X_n \mathbb{1}[|X_n| < \alpha]$ and $X^{(\alpha)} = X \mathbb{1}[|X| < \alpha]$. Prove that $X_n^{(\alpha)} \to_n X^{(\alpha)}$ a.s. for all but countably many $\alpha \in (0, +\infty)$. Conclude that $\mathbb{E}[X_n^{(\alpha)}] \to \mathbb{E}[X^{(\alpha)}]$ for all but countably many $\alpha \in (0, +\infty)$.

3. Suppose now that $\mathbb{E}[X_n] \to_n \mathbb{E}[X] < +\infty$. Prove that $\{X_n\}_{n\geq 0}$ is uniformly integrable. (Hint: use question 2. to find $\lim_{n\to\infty} \mathbb{E}[X_n \mathbb{1}[|X_n| \geq \alpha]]$ for all but countably many $\alpha \in (0, +\infty)$)

Exercise 4 (1 points). Consider the r.v. $U \sim ([0,1])$, $V = 0$ and define $\{X_n\}_{n\geq 0}$ the sequence of integrable r.v. with

$$X_n = \begin{cases} n, & \text{if } U \leq \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < U \leq 1 - \frac{1}{n} \\ -n, & \text{if } 1 - \frac{1}{n} < U. \end{cases} \tag{1}$$

Show that $X_n \to_n V$ a.s. and that $\mathbb{E}[X_n] \to \mathbb{E}[V]$. Show that $\{X_n\}_{n\geq 0}$ is **not** u.i.