PROJECTS 2 MATH 226-03, FALL 2021

RAUL PENAGUIAO

San Francisco State University

Rules and requirements

- Your project group should have between three and five people. None of your group colleagues should be the same as the ones from project 1.
- 6 sections: Introduction, Development, Examples, Conclusion, References, Pictures.
- You should hand in a report, between 4 and 7 pages without counting references and pictures.
- Your report should have plenty of pictures: these topics lend themselves very easily to draw graphs or pictures, even if it takes a bit of work to plot them in some mathematical software.
- **The projects have to be mathematical in nature**, so you should present a solution to one or several specific problems. These problems should be stated in the beginning of the report. A little bit of historical context and related problems add value.
- These are just guidelines that help you start out with a writeout about the problems, and to give some structure for the grader to have a homogeneous work. If you see a good reason to deviate from these (for instance you want to study a particular example in depth in its own section) do so.
- Agree with your group mates on a timeline for the project ahead of time.
- Due until Thursday 02nd of December, at 5pm.

E-mail address: `raulpenaguiao@sfsu.edu`.
Date: November 26, 2022.

Projects

Differential equations.

$$f'' + 2f' + f = 0$$

In this project, you will be tasked with finding solutions to several **differential equations**. A differential equation is an equality between functions and its derivatives. Often the unknown in these equations is the function. A usual example is the problem of antiderivatives, for instance the following differential equation

$$f'(x) = cos(x)$$

has a family of solutions described by $f(x) = \sin(x) + C$.

For any real number r, describe the family of solutions to the equation $f'(x) = rf(x)$. What other differential equations can you solve? Which differential equations does the function $sin(x)$ satisfy, that involves the first and the second derivative?

References: `https://www.whitman.edu/mathematics/calculus_online/section17.02.html`

Finite sums.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We have seen in class that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. We also have seen a formula for $\sum_{i=1}^{n} i^2$, that polynomialy depends on n.

In this project you are tasked to prove the formula that we found for $\sum_{i=1}^{n} i^2$, and find one such formula for $\sum_{i=1}^{n} i^3$ and $\sum_{i=1}^{n} i^4$. Can you find a pattern?

We used these formulas to estimate the integrals $\int_0^1 tdt$ and $\int_0^1 t^2 dt$, with an arbitrary number of rectangles, with the left endpoint rectangle sum. With the newfound formulas, estimate the integrals $\int_0^1 t^3 dt$ and $\int_0^1 t^4 dt$.

References: `https://www.geeksforgeeks.org/sum-cubes-even-odd-natural-numbers/`

Estimate π.

$$\pi$$

One of the oldests tasks of mathematicians was to estimate the value of π, the ratio between the radius of a circle and its circunference. After the work of Euclid, where he found exceptionally good approximations of π, interest in this task was reduced to a mathematical curiosity, as practical applications of π do not require more that a couple of digits of accuracy.

In this project you will disregard all these practical nuisances, and try to find the best approximation possible of π. You will do this by looking at integrals.

Specifically, by computing the area of a half circle by means of the integral of a suitable function, you will find that this value can be approximated arbitrarily by using a large number of rectangles.

When finding approximations of π, ensure that you actually find an interval for π. Can you make this interval have a small with of, say, 10^{-10}? What other ways can you use to approimate the value of π?

References: `https://math.stackexchange.com/questions/22777/calculate-pi-precisely-using-integrals`

The logarithm as an integral.

$$\ln(x) = \int_1^x \frac{dx}{x}$$

The natural logarithm is usually defined as the inverse of the exponential function. However, in this project we will be using a second definition of the natural logarithm:

$$\ln(x) = \int_1^x \frac{dx}{x}$$

Use this formula to estimate the value of $\ln(2)$. Prove the formula, by using the fundamental theorem of calculus.

Explore some of the known properties of the logarithm ($\ln(xy) = \ln(x) + \ln(y)$, for instance) and prove them using the new definition of ln as an integral, *i.e.* by showing that the area corresponding to $\ln(xy)$ is the sum of the areas corresponding to $\ln(x)$ and $\ln(y)$.

Finally, use this integral to prove that $\sum_{n=1}^{n} 1/n \geq \ln(n)$, and conclude that $\sum_{n=1}^{n} 1/n$ can be arbitrarily large number, for a big enough n.

`https://tinyurl.com/9bxd87cj`

Areas.

$$A_{\text{sphere}} = 4\pi r^2$$

Integration and the computation of areas have a clear relation, and we are able to estimate very accurately the area os some regions using this integration tool.

In this project, you will use the integral to compute the area of an elipse. You will also use the integral to **estimate** the area under a cycloid.

What other areas can you try and compute?

References: `https://en.wikipedia.org/wiki/Cycloid`

Volumes.

$$V_{\text{sphere}} = \frac{4\pi r^3}{3}$$

Volume of a bowl Computing volumes using integrals is easy, if you can compute the area of some planar sections of your solid.

For instance, a sphere can be seen as a collection of circles pilled together, so the volume of the sphere can be computed as

$$\int_{-1}^{1} \text{Area of circle at heigth } t \; dt$$

In this project, you will compute the volume of a bowl, assume that this bowl has a shape generated by revolving the graph of the function $f(x) = x^2/2$ around the y axis, with heigth H.

Additionally, assume that we fill the bowl at a constant rate with water. Describe the function $h(t)$ that gives the height of the volume of water with respect to time.

Solve these problems with other types of bowls, like a spherical one, or some given by other functions.

Reference: `https://tinyurl.com/2f7738bx`

Revolution solids a solid constructed via a revolution is a very common way of building solids with rotational symmetry.

In fact, these solids are so prevasive that we have a way of computing its volume with an integral formula. A couple of examples are cones, cylinders and tori. In this project you will be tasked to compute the volume of these solids using integrals.

Reference: `https://tinyurl.com/33srpxw6`, `https://math.libretexts.org/Courses/Monroe_Community_College/MTH_211_Calculus_II/Chapter_6%3A_Applications_of_Integration/6.3%3A_Volumes_of_Revolution%3A_The_Shell_Method`

Arc length.

$$L = \int \sqrt{1 + f'(x)^2} dx$$

Arc lenght computation The formula for the arc length using integrals dates back to the 17th century, and corresponds to a time in mathematics where solutions to long standing problems were booming. Specifically, the arc lenghts of elipses, catenaries and cycloids were found.

In this project, you will start by validating the formula for a simple curve, like one given by a linear function. Do you obtain the expected value?

Afterwards, you are tasked to use this formula to approximate the length of several curves, like a parabola, a half circle, etc. In some cases, you will only be able to find approximate values.

`https://en.wikipedia.org/wiki/Arc_length`

Arc lenght optimization Find a function $f(x)$ defined in the interval $[-1, 1]$ such that

- $f(-1) = f(1) = 0$.
- $\int_{-1}^{1} f(t)dt = 2$
- This function has minimal arc length.

You should try this with several different functions, defined by branches. A first guess can be the following function:

$$f(t) = \begin{cases} 2t + 2 & -1 \leq x \leq 0 \\ -2t + 2 & 0 \leq x \leq 1 \end{cases}$$

References: https://en.wikipedia.org/wiki/Arc_length

Simpson method.

$$\int_a^b f(x) \sim \sum_{i=1}^{n} f(x_i)t_i$$

The first way to approximate integrals is by using the rectangle sum, but we can quickly see that some approximations are better than others. For instance, the left endpoint rectangle sum is, in general, worse in approximating an integral than the middle point rectangle sum. This is made explicit by looking at polynomials of degree 1: the middle point rectangle sum is an exact approximation, whereas the left endpoint sum is not.

In this project you will discuss the Simpson's rule, a method used to compute the integrals of functions that is exact on all polynomials of degree two.

Can you find a rule that is exact on all polynomials of degree three? Hint: you will need to evaluate your function in at least four points.

Reference: https://en.wikipedia.org/wiki/Simpson%27s_rule