

Linear Tropical Geometry with Matroids

Discrete geometry and topological combinatorics seminar
FU Berlin

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Slides can be found at `raulpenaguiao.github.io/`
Joint work with Federico Ardila and Chris Eur

Optimization of a monomial

Fix some vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}_{>0}^n$.

Consider $f_{\mathbf{t}} : \mathbf{x} \mapsto x_1^{w_1} \dots x_n^{w_n}$ from a variety $X \subset (\mathbb{C}^*)^n$.

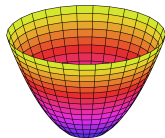


Figure: A variety where we can optimize $f_{\mathbf{w}}$

What is the number of critical points of f ? Does it depend on the choice of \mathbf{w} ? For generic \mathbf{w} , no! This number is called the **maximum likelihood degree** of a model X .

If X is a vector space, $\text{MLDeg}(X) = \beta(M)$.

Edge weight problem

Fix some vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}_{\geq 0}^n$. Can we find **compatible** weights \mathbf{x} of the edges of G , \mathbf{y} of the edges of $(G/0)^*$?

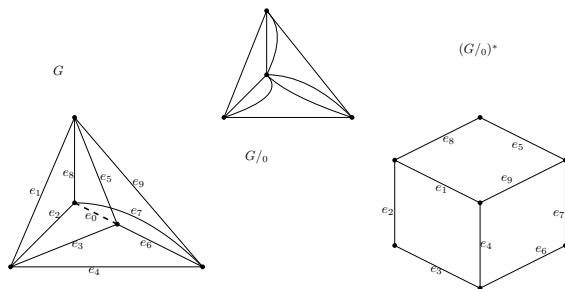


Figure: Find \mathbf{x} and \mathbf{y} edge weights that are *compatible*.

- The sum of the weights is \mathbf{w} .
- Every cycle has at least two minimal edges (edge 0 has weight 0).

Fix $\mathbf{w} = (0, 1, 1, 2, 2, 5, 3, 4, 7)$.

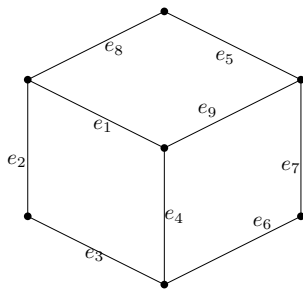
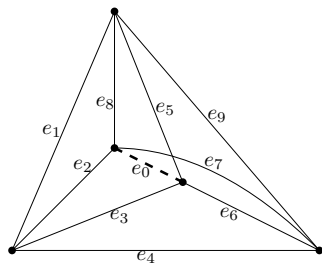


Figure: Find \mathbf{x} and \mathbf{y} edge weights that are *compatible*.

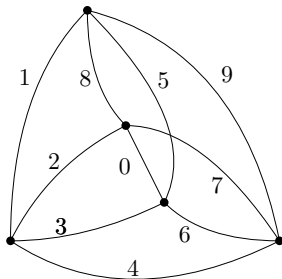
	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
\mathbf{x}	00	01	00	00	02	03	00	00	02
\mathbf{y}	000	110	111	222	20	52	33	44	75
\mathbf{w}	0	1	1	2	2	5	3	4	7

- 1 Introduction
- 2 Matroids
- 3 The Bergman Fan
- 4 Degree of Bergman Fan

Flats

Maximal sets with a fixed rank.

That is, F is a flat if for any $i \notin F$, $r_M(F \cup i) > r_M(F)$.



$\{\emptyset, \text{matchings}, \text{complete subgraphs}, \dots\}$

$\{\emptyset \subsetneq 1 \subsetneq 01 \subsetneq 0167 \subsetneq 0123456789\}$

Flats: The uniform matroid

Basis of the uniform matroid $U(n, k) =$ all sets of size k in $[n]$.

Any set of size $\leq k$ is independent.

Any set of size $\leq k - 1$ is a flat.

Any complete flag of flats is of the form

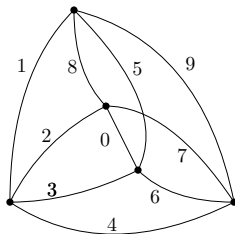
$$\{\emptyset \subsetneq \{v_1\} \subsetneq \{v_1, v_2\} \subsetneq \cdots \subsetneq \{v_1, \dots, v_{k-1}\} \subsetneq [n]\}$$

$$v_1 \mid v_2 \mid \cdots \mid v_{k-1} \mid ([n] \setminus \{v_1, \dots, v_{k-1}\})$$

Activities

Fix total order in V , ground set of a matroid M .

- $i \in B$ is internal activity if $i = \min C^\perp$, where $C^\perp \subseteq B^c \cup i$ is a cocircuit (a cut of the graph).
- $e \notin B$ is external activity if $e = \min C$, where $C \subseteq B \cup i$ is a circuit.



$$i(2567) = 0, \quad e(2567) = 2, \quad i(0146) = 2, \quad e(0146) = 1$$

Tutte polynomial

$$T_M(x, y) = \sum_{A \subseteq V} (x - 1)^{r_M(V) - r_M(A)} (y - 1)^{|A| - r_M(A)}$$

$$T_M(x, y) = \sum_{B \in \mathcal{B}} x^{i(B)} y^{e(B)} = \sum_{i,j} b_{i,j} x^i y^j$$

Observation

of bases with no external activities (called *nbc* bases):
independent of the order chosen.

of bases with no external activities and one internal activity (called β -*nbc* bases) **independent of the order chosen**

The Bergman Fan

$$\Sigma_M = \{ \vec{x} \in \mathbb{R}^n / \mathbb{R}\mathbf{1} \mid \forall C \in \mathcal{C} \text{ s.t. } \min_{c \in C} x_c \text{ attained twice} \}.$$

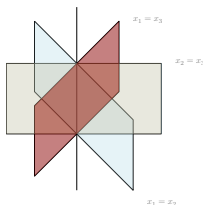


Figure: The Bergman Fan is a polyhedral fan in $\mathbb{R}^n / \mathbb{R}\mathbf{1}$.

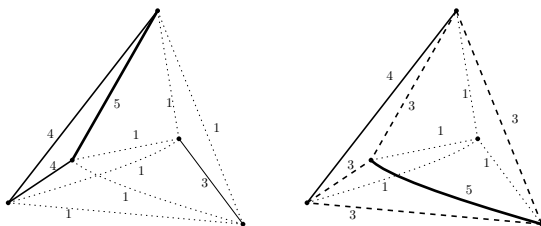


Figure: Two elements in the Bergman fan of the graphical matroid

Theorem (Sturmfels and Feichner, 2004)

The Bergman Fan of a matroid decomposes into the following cones

$$\Sigma_M = \bigcup_{\mathcal{F} \text{ flag of flats}} \mathcal{C}_{\mathcal{F}} = \bigcup_{F_1 \subset \dots \text{ flag of flats}} \{x_i \geq x_j \text{ whenever } i \in F_k, j \notin F_k\}.$$

Problem

Can we compute the degree of the tropical variety Σ_M ?

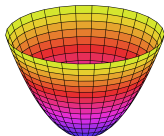


Figure: A variety: its degree is the $\#$ of intersections with a line **in the complex plane**.

Just intersect it with a hyperplane!

Note: Hyperplanes in the tropical world are Bergman fans of $U_{n,k}$.

Example of degree computation

Consider M the graphical matroid of K_5 , of rank 4 (so $\dim \Sigma_M = 3$). One has $\Sigma_M \subseteq \mathbb{R}^{10}/\mathbb{R}\mathbf{1}$, we intersect it with a hyperplane of $\dim = 6$.

$$|\Sigma_{U_{10,7}} + \underbrace{(1, 10, 100, 1000, \dots, 10^9)}_{\vec{w}} \cap \Sigma_M| = ?$$

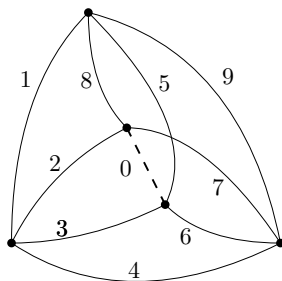


Figure: There is only one $\mathbf{x} \in \Sigma_M$ and $\mathbf{y} \in \Sigma_U$ such that $\mathbf{y} + \mathbf{w} = \mathbf{x}$. Such vector \mathbf{x} belongs to the **greedy flag of flats**.

Degree computations

Theorem (Greedy basis algorithm)

$$\deg(\Sigma_M) = 1$$

Theorem (Adiprasito, Huh, Katz, 2018)

$$\deg(-\Sigma_M) = \#\{nbc \text{ bases}\}$$

Theorem (Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, Telen 2021 and Ardila, Eur, P 2022)

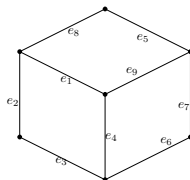
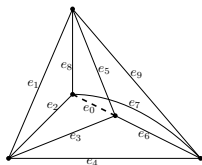
$$\deg(\Sigma_{(M/0)^\perp} \cdot -\Sigma_M) = \#\{\beta - nbc \text{ bases}\} = \beta(M)$$

Finding Nemø points form nbc bases

$$\deg(\Sigma_{(M/0)^\perp} \cdot -\Sigma_M) = ?$$

Fix generic $\mathbf{w} = (10^{i-1})_i$.

Find $\mathbf{x} \in \Sigma_M$, $\mathbf{y} \in \Sigma_{(M/0)^\perp}$ such that $\mathbf{x} + \mathbf{y} = \mathbf{w}$.



Find a β -nbc basis $B = 0257 \rightarrow 7|5|24|013689$,

corresponding β -nbc cobasis $B^\perp = 134689 \rightarrow 9|8|6|74|3|521$.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
\mathbf{x}	0	9	0	9	$10^4 - 1$	0	$10^6 - 10^3 + 9$	0	0
\mathbf{y}	1	1	100	$10^3 - 9$	1	10^5	$10^3 - 9$	10^7	10^8
\mathbf{w}	1	10	100	10^3	10^4	10^5	10^6	10^7	10^8

Biblio

- Agostini D., Brysiewicz T., Fevola C., Kühne L., Sturmfels B., Telen S. (2021). *Likelihood Degenerations*. [Motivation behind ML degree computations](#)
- Adiprasito K., Huh J., Katz E. (2018). *Hodge Theory for combinatorial geometries* [Degree computations in matroids](#)
- Ardila F., Eur C., RP (2022) *The maximum likelihood of a matroid*.

Thank you

