Introduction to Tropical Combinatorics with Matroids

Seminar Algebra and Combinatorics, University of Leipzig

Raul Penaguiao

MPI MiS Leipzig

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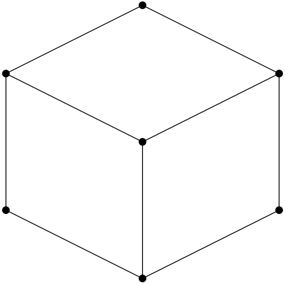
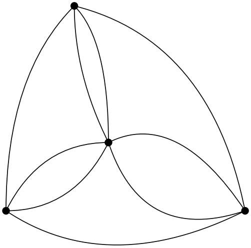
Slides can be found at `raulpenaguiao.github.io/`
Joint work with Federico Ardila and Chris Eur

Maximum Likelihood problem

Consider a monomial function $f_{\mathbf{t}} : \mathbf{x} \mapsto x_1^{t_1} \dots x_n^{t_n}$ from a variety $X \subset (\mathbb{C}^*)^n$, defined for some vector $\mathbf{t} \in \mathbb{Z}^n$.
It turns out that the number of critical points of f does not depend on the choice of $\mathbf{t}$. This is called the **Maximum Likelihood degree** of a model X.

Edge weight problem





- Corresponding edges have the same value.
- The sum of the weights is t.
- No cycle has a unique minimal edge.
- Fix one edge to have weight zero.

Matroids

Defines a notion of **independence**. Abstract notion of sets of vectors.

Definition

A matroid M is a pair $(V, \mathcal{B})$ such that $B \in \mathcal{B}$ are subsets of V, called ***bases****, satisfy*

- $\mathcal{B} \neq \emptyset$
- *For any $B, B' \in \mathcal{B}$, $i \in B \setminus B'$ there is some basis $j \in B' \setminus B$ such that $B \Delta \{i, j\}$ is a basis of M.*

Matroids: Examples

The **Uniform matroid** $U_{n,k}$ on n elements of degree k is a matroid of the form $([n], \binom{[n]}{k})$.

The graphical matroid (edges, spanning forests)

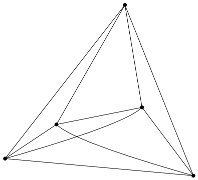


Figure: To a graph it corresponds a graphical matroid.

Matroids: Examples

Given a collection of vectors $(\vec{v}_i)_{i \in V}$, a matroid in V is given by maximal subcollections of l.i. vectors.

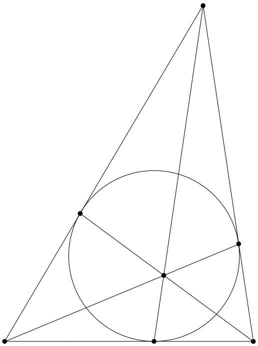


Figure: Not all matroids are representable.

Circuits

Circuits are minimal dependent sets.

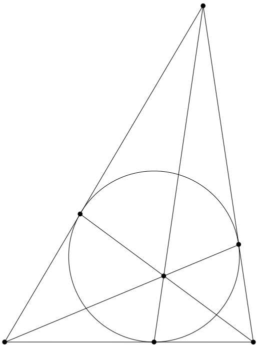
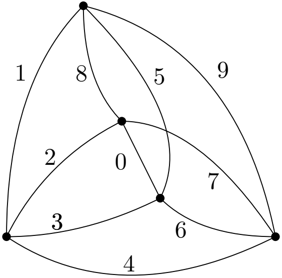


Figure: The circuits are all the lines, along with some four-element sets.

Rank function

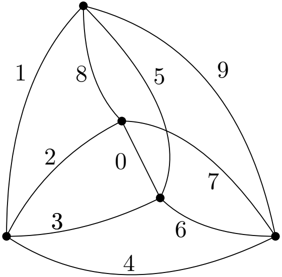
$$r_M(A) = \max_{I \text{ independent}} |A \cap I| .$$



$r(1238) = 3$ $\quad$ $r(57) = 2$ $\quad$ $r(\emptyset) = 0$

Flats

Maximal sets with a fixed rank.
That is, F is a flat if for any $i \notin F$, $r_M(F \cup i) > r_M(F)$.



$\{\emptyset,$ matchings , complete subgraphs , ...$\}$

$$\{\emptyset \subsetneq 1 \subsetneq 01 \subsetneq 0167 \subsetneq 0123456789\}$$

Flats: The uniform matroid

Basis of the uniform matroid $U(n, k)$ = all sets of size k in $[n]$.
Any set of size $\leq k$ is independent.
Any set of size $\leq k - 1$ is a flat.
Any complete flag of flats is of the form

$$\{\emptyset \subsetneq \{v_1\} \subsetneq \{v_1, v_2\} \subsetneq \cdots \subsetneq \{v_1, \ldots, v_{k-1}\} \subsetneq [n]\}$$

$$v_1 \,|\, v_2 \,|\, \ldots \,|\, v_{k-1} \,|\, ([n] \setminus \{v_1, \ldots, v_{k-1}\})$$

Recap

- Bases = maximal independent sets
- Circuits = minimal dependent sets
- Flats = Maximal sets of a given rank

Dual matroids

If $M = (V, \mathcal{B})$ is a matroid, $M^{\perp} = (V, \{B^c | B \in \mathcal{B}\})$ is its dual.
The dual of a graphical matroid is the graphical matroid of the dual graph.

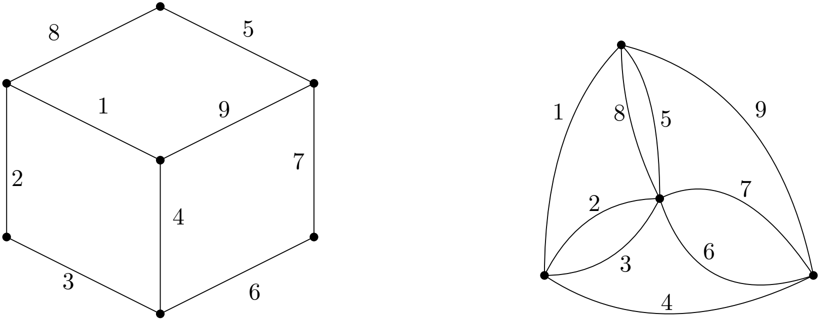
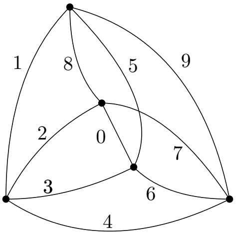


Figure: A graph and its dual. The corresponding matroids are dual. Can you completely describe the cocircuits of a graph?

Activities

Fix total order in V, ground set of a matroid M.

- $i \in B$ is internal activity if $i = \min C^{\perp}$, where $C^{\perp} \subseteq B^c \cup i$ is a cocircuit.
- $e \notin B$ is external activity if $e = \min C$, where $C \subseteq B \cup i$ is a circuit.



$$i(2567) = 0, \;\; e(2567) = 1, \;\; i(0146) = 2, \;\; e(0146) = 1$$

Tutte polynomial: Example

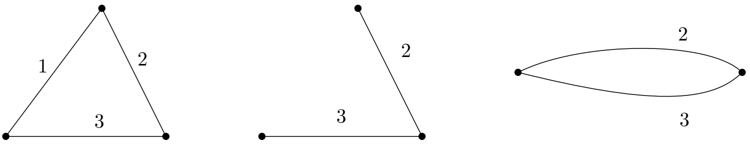


Figure: A graphical matroid M, and its deletion and contraction on the edge 1.

$$T_M(x,y) = \sum_{A \subseteq V} x^{r_M(V) - r_M(A)} y^{|A| - r_M(A)}$$

$$T_{M \backslash 1}(x,y) = 1 + 2x + x^2$$

$$T_{M/_1}(x,y) = 2 + x + y$$

$$T_M(x,y) = 3 + 3x + x^2 + y$$

Tutte polynomial: Deletion-contraction

$$T_M(x,y) = \sum_{A \subseteq V} x^{r_M(V) - r_M(A)} y^{|A| - r_M(A)}$$

Deletion-contraction invariant if e is not loop nor coloop:

$$T_M(x,y) = T_{M \setminus e}(x,y) + T_{M/_e}(x,y)$$

Tutte polynomial: counting bases

$$T_M(x-1, y-1) = \sum_{B \in \mathcal{B}} x^{i(B)} y^{e(B)} = \sum_{i,j} b_{i,j} x^i y^j$$

Observation

The number of bases with no external activities (called nbc bases) ***is independent of the order chosen***

The number of bases with no external activities and one internal activity (called β-nbc bases) ***is independent of the order chosen***

The Bergman Fan

$$\Sigma_M = \{\vec{x} \in \mathbb{R}^n/_{\mathbb{R}\mathbb{1}} | \forall_{C \in \mathcal{C}} \text{ s.t. } \min_{c \in C} x_c \text{ attained twice } \}.$$

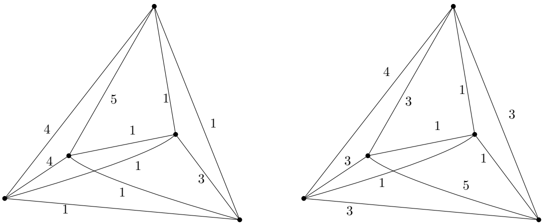
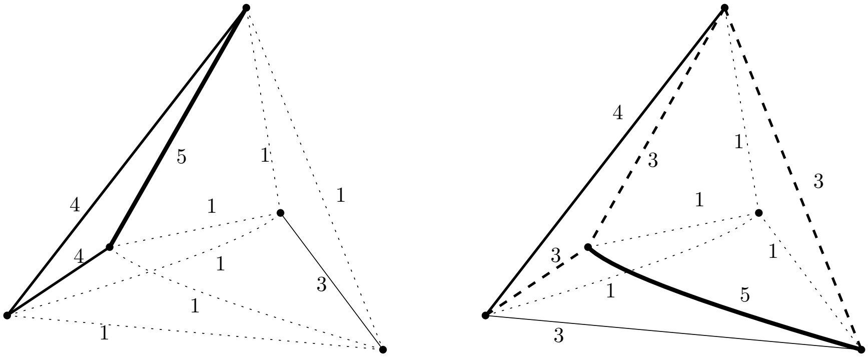


Figure: Two elements in the Bergman fan of the graphical matriod



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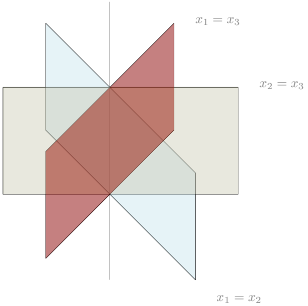


Figure: The Bergman Fan is a polyhedral fan

Theorem (Sturmfels and Feichner, 2004)

The Bergman Fan of a matroid decomposes into the following cones

$$\Sigma_M = \bigcup_{\mathcal{F} \text{ flag of flats}} \mathcal{C}_{\mathcal{F}} = \bigcup_{F_1 \subset \dots \text{ flag of flats}} \{x_i \geq x_j \text{ whenever } i \in F_k, j \notin F_k\}\,.$$

Problem

Can we compute the degree of the tropical variety Σ_M?

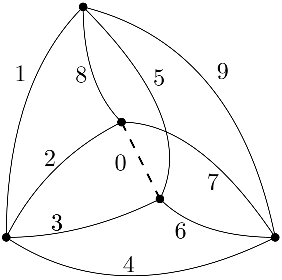
Just intersect it with a plane!

Planes in the tropical world are Bergman fans of the matroid $U_{n,k}$

Example of degree computation

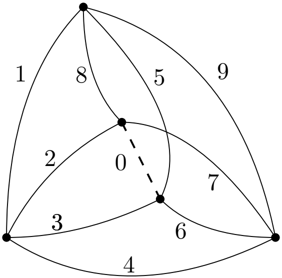
Consider M the graphical matroid of K_5, of rank 4. One has $\Sigma_M \subseteq \mathbb{R}^{10}/_{\mathbb{R}\mathbb{1}}$

$$|\Sigma_{U_{10,7}} + \underbrace{(1, 10, 100, 1000, ..., 10^9)}_{\vec{\omega}} \cap \Sigma_M| = ?$$



$$\vec{v} = (10^6, 10^4, 10^4, 10^4, 10^4, 10^6, 10^6, 10^8, 10^8, 10^9)\,.$$

$\vec{v}$ belongs to the cone $\mathcal{C}_{9|87|650|1234} \subseteq \Sigma_M$
and $\vec{v} - \vec{\omega}$ belongs to $\mathcal{C}_{7|0|5|1|2|3|4689} \subseteq \Sigma_{U_{10,7}}$



$\vec{v}$ belongs to the cone $\mathcal{C}_{9|87|650|1234} \subseteq \Sigma_M$
and $\vec{v} - \vec{\omega}$ belongs to $\mathcal{C}_{7|0|5|1|2|3|4689} \subseteq \Sigma_{U_{10,7}}$
where $\vec{\omega} = (1, 10, \ldots, 10^9)$.

Degree computations

Theorem

$$\deg(\Sigma_M) = 1$$

Theorem (Adiprasito, Huh, Katz, 2018)

$$\deg(-\Sigma_M) = \#\{\textit{nbc bases}\}$$

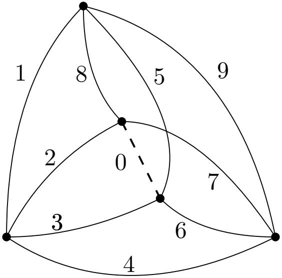
Theorem (Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, Telen 2021 and Ardila, Eur, P 2022)

$$|\Sigma_{(M\backslash 0)^\perp} \cap_{st} -\Sigma_M| = \#\{\beta - \textit{nbc bases}\}$$

Examples

If M is the graphical matroid

$$|\left(\Sigma_{U_{10,7}}+\vec{\omega}\right)\cap\Sigma_M|$$



Then the unique intersection point is

$$\vec{v}=(10^6,10^4,10^4,10^4,10^4,10^6,10^6,10^8,10^8,10^9)\,.$$

Theorem

The unique intersection point of

$$\Sigma_{U_{n,n-r+1}} + \vec{\omega} \cap \Sigma_M$$

lies in $\mathcal{C}_{\mathcal{F}(B)}$, the cone corresponding to the greedy basis with respect to the order induced by $\vec{\omega}$.

Theorem

Each intersection point in

$$|\Sigma_M \cap (\omega - \Sigma_{U_{n,n-r+1}})$$

lies in $\mathcal{C}_{\mathcal{F}(B)}$, the cone of Σ_M corresponding to an nbc-basis with respect to the order induced by $\vec{\omega}$.

Theorem

Each intersection point in

$$|\Sigma_M \cap (\omega - \Sigma_{(M\backslash 0)^\perp})$$

lies in $\mathcal{C}_{\mathcal{F}(B)}$, the cone of Σ_M corresponding to a β-nbc-basis with respect to the order induced by $\vec{\omega}$.

Biblio

- Agostini D., Brysiewicz T., Fevola C., Kühne L., Sturmfels B., Telen S. (2021). *Likelihood Degenerations*. Motivation behind ML degree computations
- Adiprasito K., Huh J., Katz E. (2018). *Hodge Theory for combinatorial geometries* Degree computations in matroids
- Ardila F., Eur C., RP (2022) *The maximum likelihood of a matroid* All these computations are here! To appear

Thank you